Frost was, as R. Tucker writes, a great favorite at Cambridge, a sure concomitant of his brightness, cheerfulness, kindness of heart and consideration for others.

His exceptional life-long experience gives tremendous weight to his testimony in favor of the early cultivation of a talent for mathematics.

He explicitly says: "To attempt after a certain age to acquire ease in mathematical operations is like a grown man trying to learn the violin."

SOME REMINISCENCES IN REGARD TO SOPHUS LIE.

By DR. G. A. MILLER.

Since the death of the famous Norwegian mathematician, Sophus Lie, a large number of appreciative articles on his life and work have appeared in the mathematical journals of various countries. Two such articles were published in the April number of this journal. In what follows we shall endeavor to add a few details (based upon personal observations) which may assist those who never met him to form a more accurate idea with respect to him.

The trait of Lie's character which impressed me most forcibly when I first met him in the summer of 1895 was his extreme openness and lack of effort to hide ignorance on any subject. It is well known that he began the study of discontinuous groups under the masterly guidance of Sylow. He maintained a deep interest in these groups, making frequent use of Jordan's classical "Traité des Substitutions," and he spoke in terms of great respect of the work of Jordan, Frobenius, and others who were working in the field of discontinuous groups.

He, however, never felt at home in this subject. In fact he frequently remarked during his lectures that he always got stuck when he entered upon the subject of discontinuous groups, To him the continuous groups seemed more simple as well as more useful. He frequently used the expression, "the discontinuous groups are good but my (continuous) groups are better," and he advised his students to begin their study of groups with the continuous groups and to take up the study of the discontinuous groups later.

He was an inspiring teacher but his lectures were not always well prepared. Sometimes he had to pay quite heavily for this lack of preparation, being unable at the moment to prove simple things in his own theory. It was an interesting sight to see him at the board working away with all his might and calling on his students to help him out of the difficulty, using the expression, "Here I stick, will not one of you help me out." He kept in good spirits at such occasions but he generally could not maintain enough self-possession to work his way out of the difficulties during the rest of the hour.

He was somewhat careless in regard to his dress. If he felt uncomfort-

able he would take off his collar and tie before his class, lay them aside, and then go on with his lecture. At one time the room happened to be somewhat cold so he put on his big hat and kept in on during the entire lecture. He considered his lectures as gymnastics for the students, being bent on training them to think according to his methods rather than on giving them a systematic treatise on any subject.

In his intercourse with the students he was friendly and approachable. He seemed to pay special attention to foreign students, claiming that they were generally better prepared than his German students. He would talk for hours with foreign students, asking questions in regard to their work as well as in regard to the mathematicians of their country. In case a student was well acquainted with any mathematical subject he would ask questions in regard to details which were unknown to him, candidly stating that he was asking for information.

During the summers of 1895 and 1896 he returned from his summer vacation about one month before the opening of the university and lectured daily before the few American students of mathematics who happened to be in Leipzig, with a view to prepare them to follow his lectures better during the university year. He did this work gratis and at his own suggestion. These lectures were very elementary and frequently took the form of a colloquium, the aim being to make the student familiar with the mathematical phrases rather than to teach much mathematics.

An idea of his view of mathematics in general may be inferred from the following incident. The building in which he held most of his lectures was royal property and the king of Saxony lived in a part of it whenever he came to Leipzig. One day Lie remarked to a few Americans, "Perhaps you do not know that our rooms are in the royal palace. It seems to me very appropriate that mathematics should be taught in a royal palace for it certainly is a royal subject. In some of the other departments one generation discards what the preceding generation regarded as established facts, but in mathematics an established fact remains valuable forever. We have just as much admiration for mathematical facts that were proved 2000 years ago as for those which have been established during our generation.

"Mathematics is a royal subject but the trouble is that the mathematicians do not get royal pay. Some of the professors here who lecture on popular subjects get several thousand dollars per year in students' fees, while the mathematician gets scarcely anything beyond his regular salary. In the popular departments the advanced subjects are frequently assigned to the younger men while the professors lecture on the elementary subjects in order to increase their income. In mathematics we are not tempted to commit this sin since all of our classes are small."

Lie was regarded as one of the easiest men at Leipzig for the doctor thesis. He generally assigned easy subjects and he assisted the students very freely. He was a hard worker, working seven days per week. He used to say

that he was almost certain to make mistakes in going over a piece of work for the first time, and that he could generally not find his errors until he would work over the problem a second time using different symbols. He frequently got very enthusiastic over the theory of groups remarking (not very seriously) that everything could be done by means of groups.

The students talked a great deal about his peculiarities but they had great respect for his attainments. In speaking about Klein and Lie, a Frenchman who had studied under both remarked, "Klein is a gentleman while Lie is just a good fellow." I presume many of those who studied under both would agree that this statement conveys a great deal of truth.

I feel encouraged to publish these few personal observations because I believe that Lie would have liked to have his faults go with his merits. I do not think he wanted to be regarded as one who thought that he had mastered every part of the extensive science of mathematics or as one who thought that all his habits were exemplary. He believed that he had contributed materially towards the advancement of the science of mathematics and he worked hard to accomplish this end. Future developments along the lines which have been emphasized or opened by him will have a great influence on his relative position among the mathematicians of this century.

Cornell University, June, 1899.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

Grassmann's *Ausdehnungslehre* is one of the few great works of mathematics of the 19th century. Appearing first in 1844 and rewritten in 1862, it is only within the last decade or two that it has received a tardy recognition. One reason for this is found in the difficulty of the subject itself, being unlike other mathematics; and another, in the rigorous methods of presentation adopted by the author. In the *Ausdehnungslehre* of 1862, following some 150 pages of theory, the author for the first time gives the subject concrete form by applying his method to geometry. The theoretical part is naturally the more difficult, while the application to geometry is the more interesting. Hyde, in his *Directional Calculus*, purposing to present the *Ausdehnungslehre* to American readers, cut the knot of the difficulty by taking the results of the theoretical part for granted and giving only the application to geometry, and by limiting his treatment to two and three dimensional space.

An elementary exposition which will give the simpler portions of the theoretical part as well as the applications of the theory seems to be needed.

Such an exposition should serve the needs of two classes of readers : First, of those who would like to have a good general idea of the subject without going very deeply into its particulars ; and secondly, of those who, expecting to make a thorough study of the subject, wish first to read an introduction to it. To meet this want the following pages have been written. In some places for the purpose of making the subject clearer, changes have been made, but wherever they have been introduced, attention is always called to them.

CHAPTER I.

INTRODUCTION.

1. In elementary mathematics only one kind of unit is admitted, or at most two, viz., 1 and $\sqrt{-1}$. In the Theory of Extension besides the absolute unit of arithmetic and algebra, *extensive quantities* appear. The extensive quantities are different in nature from the absolute unit and different from each other. As a simple example of an extensive quantity we may name a *vector* (§3).

2. A *Scalar* quantity is a quantity of elementary mathematics, *i. e.*, a simple number, either positive or negative.

3. *A Vector* is a straight line whose length and direction are fixed but not its position. Thus any two parallel and equal straight lines may represent the same vector. A vector gives the relative position of one point with reference to another, viz., a certain distance in a certain direction.

4. Vectors can be *added* and *subtracted*.

Thus if ε_1 is the vector from O to A and ε_2 the vector from A to B, the sum of ε_1 and ε_2 is ε_3, because translation from O to B along the straight line OB is equivalent, or equal in the vector sense, to translation along OA and AB. Thus,

$$\varepsilon_1+\varepsilon_2=\varepsilon_3.$$

Transposing, we get

$$\varepsilon_1=\varepsilon_3-\varepsilon_2=\varepsilon_3+(-\varepsilon_2).$$

Interpreting this equation we see that translation along ε_3 followed by translation along ε_2 *in the negative direction* is equal to translation along ε_1.

The sum of any number of vectors may evidently be found in the same way. Thus, in the figure

$$\varepsilon_5=\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4.$$

Stated generally, we have

The sum of any number of vectors is found by joining the beginning point of the second vector to the end point of the first, the beginning point of the third to the

end point of the second, and so on ; the vector from the beginning point of the first vector to the end point of the last is the sum required.

The sum and difference of two vectors are the diagonals of the parallelogram whose adjacent sides are the given vectors.

Or, more explicitly—

(1). *The sum of two vectors going out from an origin and forming two sides of a parellelogram is that diagonal of the parallelogram which passes through the origin.*

(2). *The difference of two such vectors is that diagonal which proceeds from the end of the subtrahend vector to the end of the minuend vector.*

5. Vectors and line segments give us simple examples of extensive quantities. We proceed to show how lines and vectors can be used in a system of coördinates.

6. The simplest case of this is where a point is located on a given line. Let ρ denote any given line, and let O be an origin on it. Let further x be a scalar. Then by giving the proper value to x, $x\rho$ will locate any point P whatever on the line. Here ρ is to be regarded as an extensive quantity since it denotes not a number but the position and length of a line.

7. The next simplest case of coördinates is that in which a point is located in a plane by means of two vectors.

Let O, the origin, be a point in the given plane, and ε_1 and ε_2 two unit vectors in this plane. Then, by making

$$\rho = x_1\varepsilon_1 + x_2\varepsilon_2$$

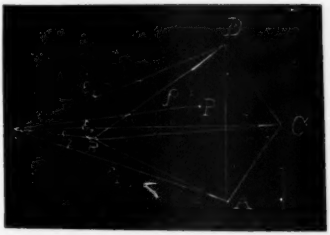
$$\text{where } x_1 = \frac{r \sin BOP}{\sin BOA}, \quad x_2 = \frac{r \sin POA}{\sin BOA},$$

and r=length of ρ, the point P may be located at any point of the plane. Here ε_1, ε_2 and ρ are extensive quantities, and x_1 and x_2 are scalars.

8. In space we may have a similar system containing three vectors. Thus, if

$$\rho = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3$$

by assigning values to x_1, x_2, and x_3, P the extremity of ρ may be located at any point in space.



9. In the same way we can have a system including four vectors. Thus, if P is any point, $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ are four unit vectors, and

$$x_1 = \frac{\text{pyramid } P-BDC}{\text{pyramid } A-BDC}, \quad x_2 = \frac{\text{pyramid } P-ADC}{\text{pyramid } B-ADC}$$

etc., we have the equation

$$\rho = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4,$$

i. e. translation along ε_1 a distance equal to x_1, followed by translation along ε_2 a distance equal to x_2, and so on, is equivalent to translation from O to P direct. A geometrical proof of the truth of this can be given, but it is not thought necessary to insert it here.

REMARK. In the preceding the ε's in a certain sense denote dimensions. Then the *space* considered in this last article is of the *fourth* order.

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF EXTENSIVE QUANTITIES.

10. DEFINITION. A quantity is said to be *independent* when it can not be expressed in terms of others. A quantity is said to be *dependent* when it can be numerically expressed in terms of others, *i. e.* as a sum formed out of numerical multiples of these quantities.

Thus if $a_1, a_2, \ldots\ldots$ are extensive quantities, and $\alpha_1, \alpha_2, \ldots\ldots$ are real numbers, positive or negative, and

$$a = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \ldots\ldots$$

a is said to be a dependent quantity and to be '*numerically expressed*' in terms of $a_1, a_2, a_3, \ldots\ldots$

11. A quantity a_1 is a *Unit* if it can serve along with other like units, $a_2, a_3, \ldots\ldots$ to give a series of numerically derived quantities, $a\ldots\ldots$ A unit is said to be *original* if it is not derived from other units. A set of quantities which are independent, *i. e.*, no one of which is numerically expressible in terms of one or more of the others is called a *System of Units*, provided any number of other quantities can be expressed in terms of them.

As an illustration of such a system we may take the set of units given in Art. 8. There $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are a set of quantities which are independent because any sum formed out of multiples of ε_1 and ε_2 can never be a quantity like ε_3, since any sum formed from ε_1 and ε_2 would be a quantity in the plane of these two (7) while ε_3 is outside of this plane. Moreover, any number of other quantities, ρ's, can be derived from $\varepsilon_1, \varepsilon_2, \varepsilon_3$.

12. DEFINITION. An Extensive *Quantity* is a quantity numerically derived from a *system* of units. If an extensive quantity can be derived from the original units it is called an extensive quantity of the *first kind*.

13. DEFINITION. Quantities from the same system can be added (or subtracted) by adding the numerical coefficients of the same units.

Thus

$$\Sigma\alpha e+\Sigma\beta e=\Sigma(\alpha+\beta)e$$

where the α's and β's under the summation signs are numbers, and the e's are extensive units. We may add here that in the *Ausdehnungslehre* the distributive law is always assumed to hold.

REMARK. To remove ambiguity it will be understood that all indicated operations are performed as one comes to them from the left. Thus $a+b+c$ means $(a+b)+c$, and abc means $(ab)c$.

14. The following formulas underlie and justify all the operations involved in addition and subtraction in algebra. They follow directly from the definition in 13.

(1). $a+b=b+a$, a commutative law in addition and subtraction.

(2). $a+(b+c)=a+b+c$, associative law in addition and subtraction.

(3). $a+b-b=a$ }
(4). $a-b+b=a$ } *opposite* character of addition and subtraction.

Hence all the laws for addition and subtraction of algebraic numbers hold also for extensive quantities.

15. DEFINITION. When an extensive quantity is multiplied (or divided) by a *number* each of its coefficients is multiplied by that number.

Thus

$$\Sigma\alpha e.\beta=\Sigma(\alpha\beta)e.$$

REMARK. If a is an extensive quantity and α a number, then in αa or $a\alpha$ the numerical factor is the multiplier and the other factor is the multiplicand.

16. From Art. 15 we infer the following formulas :

(1). $a\alpha=\alpha a$,
(2). $a\beta\gamma=a(\beta\gamma)$,
(3). $(a+b)\gamma=a\gamma+b\gamma$,
(4). $a(\beta+\gamma)=a\beta+a\gamma$,

where, as heretofore, the Greek letters denote real numbers and the Roman letters, extensive quantities. From these formulas it follows—

That all the laws of multiplication and division of algebraic quantities hold also for extensive quantities multiplied or divided by numbers.

17. DEFINITION. The totality of quantities which are derivable from a series of extensive quantities, a_1, a_2, a_3, ... a_n is called the *Space* of those quantities. A space which can be formed out of not less than n such quantities each of the first kind (12) is called a space of the nth order.

18. DEFINITION. If every quantity of a space (A) is at the same time a quantity of another space (B), while the converse is not true, then the spaces are said to be *incident;* the first is said to be *subordinate* to the second, and the second, to *include* the first.

19. *If n independent quantities a_1, ... a_n can be numerically expressed in*

terms of n other quantities $b_1 \ldots\ldots b_n$, *then is the space of the first quantities identical with that of the last quantities. But if the n quantities* $a_1 \ldots\ldots a_n$ *can be expressed in terms of less than n quantities* $b_1 \ldots\ldots b_n$, *then* $a_1 \ldots\ldots a_n$ *are not independent, and some of them can be numerically expressed in terms of others.*

20. *Two quantities of a space of the nth order are equal to each other when and only when their numerical coefficients of the same units are equal.* This is analogous to the algebraic theorem which says that two complex numbers are equal only when their real parts are equal and also their imaginary parts.

21. *If the coefficients* $x_1 \ldots\ldots x_n$ *by which an extensive quantity* x *is expressed in terms of the units* $e_1 \ldots\ldots e_n$ *satisfy an equation of the mth degree* $f(x_1 \ldots\ldots x_n)=0$, *then the coefficients* $y_1 \ldots\ldots y_n$ *by which* x *is expressed in terms of* $a_1 \ldots\ldots a_n$ *of the same space also satisfy an equation of the mth degree, and if the first equation is homogeneous, the latter is also.*

PROOF. Let $a_1=\Sigma\alpha_{1r}e_r, \ldots\ldots$ Then we have

$$x_1e_1+x_2e_2+\ldots\ldots+y_1\Sigma\alpha_{1r}e_r+y_2\Sigma\alpha_{2r}e_r+\ldots\ldots=\Sigma y_r\alpha_{r1}.e_1+\Sigma y_r\alpha_{r2}.e_2+\ldots\ldots$$

$$\therefore\ x_1=\Sigma y_r\alpha_{r1},\ x_2=\Sigma y_r\alpha_{r2},\ \ldots\ldots\quad \text{(Art. 20)}.$$

But if these values are substituted in $f(x_1, \ldots\ldots x_n)=0$, we get an equation of the mth degree in $y_1, y_2, \ldots\ldots$, and, indeed, homogeneous if the first equation is homogeneous.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

112. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is $\$4.\frac{.\dot{2}9\dot{7}}{1.00\dot{2}}$. The selling price is $\$6.\frac{1000}{33337}$. What is the gain %?

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville Tenn., and the PROPOSER.

$.\dot{2}9\dot{7}=\frac{297}{999}=\frac{11}{37}$; $1.00\dot{3}=1\frac{1}{300}=\frac{301}{300}$.

$\frac{11}{37}\div\frac{301}{300}=\frac{3300}{11137}$; $\therefore\frac{3300}{11137}=\frac{300}{1113\frac{7}{11}}$.

$\therefore\ \$4.\frac{297}{1.003}=\$4\frac{330}{111337}=\$\frac{444678}{111337}$=cost price.

$\$6.\frac{1000}{333337}=\$6\frac{1998}{333337}=\$\frac{2001332}{333337}$=selling price.

$\therefore\ \$\frac{2001332}{333337}-\$\frac{444678}{111337}=\$\frac{1556654}{1053443}$=gain.

$(\frac{1556654}{1053443}\div\frac{444678}{111337})$ of $100\%=48\frac{1044444}{2011178}\%$.

Also solved by *O. S. WESTCOTT* and *ALOIS F. KOVARIK*.

113. Proposed by B. F. SINE, Principal of Normal School, Capon Bridge, W. Va.

In what time can a note of \$5280, bearing 6% interest, be paid by paying \$600 a year? [Solve by arithmetic].

Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Let Q be the principal, r the rate, n the number of years, and P annual payment. Then

$Q(1+r)$=amount due at end of first year.

$Q(1+r)-P$=principal to run second year.

$Q(1+r)^2-P(1+r)$=amount due at end of second year.

$Q(1+r)^2-P(1+r)-P$=principal to run third year.

. . . . . .

$Q(1+r)^n-P(1+r)^{n-1}\ldots\ldots P(1+r)-P$=amount to run $(n+1)$th year.

But the debt is cancelled. Hence

$$Q(1+r)^n-P(1+r)^{n-1}-\ldots\ldots P(1+r)-P=0.$$

$$\therefore P\left[\frac{(1+r)^n-1}{r}\right]=Q(1+r)^n.$$

$$\therefore (P-Qr)(1+r)^n=P.$$

$$\therefore n=\frac{\log P-\log(P-Qr)}{\log(1+r)}.$$

In the problem $P=\$600$, $Q=\$5280$, $r=.06$.

$$\therefore n=\frac{\log 600-\log(600-5280\times.06)}{\log(1.06)}=12.88 \text{ years}.$$

Also solved by *G. B. M. ZERR*, *COOPER D. SCHMITT*, and *J. SCHEFFER*.

ALGEBRA.

93. Proposed by CHARLES C. CROSS, Whaleyville, Va.

Given $x^x+y^y=285$, and $y^x-x^y=14$, to find the values of x and y. [From *Bonnycastle's Algebra*, 1841].

I. Solution by A. H. BELL, Hillsboro, Ill.

The two equations give

$$(285-y^y)^y=(y^x-14)^x\ldots\ldots(1).$$

In (1), assuming y equal to 3, 2.9, and 2.8, by logarithms, $y=2.8248$ nearly, $=\log 0.450987$.

Multiplying by y, $\log(y^y)=\log 1.273952, =18.791=y^y$.

But $x^x=285-y^y$.

$\therefore\ 285-18.791=266.791, =x^x$.

In a table of logarithms by $\log x+\log.$ of $\log x$, x is found to be nearly $4.01637+$.

II. Solution by J. M. BOORMAN, Consultative Mechanician, Etc., Woodmere, N. Y.

$x=4.016631(8\pm$; $x_1=4.04444(4\pm$; $x_2=4.045\pm$;* $x_a<0=-?$;

$y=2.8157(12\pm$; $y_1=-1.922(77\pm$; $y_2=-1.92\pm$; $y_a>0=+?$;

curiously enough always a quartic (even if $x=6$ or more, $y=5$ or more).

*Because $x=v$ nearly in $v^v=285$, as if $y<0$. $\therefore\ y^y=(1/y)^n$; $x^y=(1/x)^n=$ fractions. $\therefore$ as $v=4.0449$ nearly, x_1 (must) <4.0449, x_2 (must) >4.0449, so $y_2{}^{x_2}$ (must)$=14.069$ nearly by (2), as $x_2{}^{y_2}=$a negative fraction$=0.069\pm$. Solve $10^{x\log x}+10^{y\log y}=10^{\log 285}$.....(1), and $10^{x\log y}-10^{y\log x}=10^{\log 14}$ .....(2), by double position. Note that $10^x=10^{4016631}=4016631$ power,—four millionth, etc., power of the millionth root of 10. So, curiously, we must compute to an even decimal if $x>0$, and to an odd decimal digit if $x<0$, to get true results.

III. Solution by OLIVER S. WESTCOTT, Principal North Division High School, Chicago, Ill.

I know of no direct method of solution, and as *Bonnycastle's Algebra*, from which it was taken, gives some idea of the use of logarithms and also of the rule known to arithmeticians in the days of Daboll as Double Position, it is presumable at least, that Bonnycastle's method was something like the following:

If x were 4 and y were 3, we should have $4^4+3^3=283$ and $3^4-4^3=17$.

Evidently then, $x>4$ and $y<3$.

Put $y=2.75$. Then $x^x+2.75^{2.75}=285$, and $x^x=268.85028$.

$x\log x=\log 268.85028$, $x=4.0205$.

Substituting these values of x and y in equation (2) we have

$$(2.75)^{4.0205}-(4.0205)^{2.75}=12.494296. \quad 14-12.494296=1.50570 \text{ error.}$$

Again, put $y=2.85$. Then $x^x+(2.85)^{2.85}=285$. $x^x=265.21628$. $x\log x=\log 265.21628$. $x=4.0148$.

Substituting as before,

$$(2.85)^{4.0148}-(4.0148)^{2.85}=14.471434. \quad 14.471434-14=0.471434 \text{ error,}$$

$$\text{and } \frac{0.471434\times 2.75+1.505704\times 2.85}{1.505704+0.471434}=2.82615$$

for an approximate value of y.

Since 2.85 is plainly much nearer the result sought for than 2.75, put $y=2.83$.

Then $x^x+(2.83)^{2.83}=285$. $x^x=266.00867$. $x\log x=\log 266.00867$. $x=4.01605$.

Substituting in equation (2) as before $(2.83)^{4.01605}-(4.01605)^{2.83}=14.083487$. $14.083487-14=0.083487$ error. Then

$$\frac{0.471434\times 2.83-0.083487\times 2.85}{0.471438-0.083487}=2.825696.$$

Bonnycastle's answer for y is 2.825716.

Again $x^x+(2.825696)^{2.825696}=285$. $x^x=266.17636274$. $x\log x=\log 266.17636274$. $x=4\ 016348$.

Bonnycastle's answer for x is 4.016698.

These trifling discrepancies may easily be occasioned by variation in the logarithmic tables used. I have made use of Chambers's seven place tables.

I have not succeeded in effecting a neat and satisfactory solution; by trial, however, I find, approximately, $x=3.89$, $y=3.54$. G. B. M. ZERR.

I do not believe a strictly algebraic solution of this problem can be given, because it is not an algebraic, but a transcendental function. Algebraically there are really four unknowns, x, $\log x$, y, and $\log y$. Of course, $\log x$ and $\log y$ are known if x and y are known, but they are here algebraically separate unknowns because of the manner in which they enter the equations. W. H. DRANE.

GEOMETRY.

---

114. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

If a variable ellipse hyperosculate a fixed ellipse at the extremity of the minor axis, the locus of the foci is a circle whose diameter is equal to the radius of curvature.

Solution by the PROPOSER.

Let $x^2/a^2+y^2/b^2=1$.....(1) be the given ellipse; then the tangent at the vertex of the minor axis is $y/b-1=0$.....(2), and the variable conic is

$$x^2/a^2+y^2/b^2-1-\lambda[(y/b)-1]^2=0.....(3), \text{ or}$$

$$x^2/a^2+(1-\lambda)(y^2/b^2)+(2\lambda/b)y-(\lambda+1)=0\ .....(4).$$

Now, the foci of the general conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.....(5),$$

are given by

$$C(x^2-y^2)+2Fy-2Gx+A-B=0.....(6),$$

$$Cxy-Fx-Gy+H=0 \ldots\ldots (7),$$

in which $A=bc-f^2$, $B=ca-g^2$, $C=ab-h^2$, $F=gh-af$, $G=hf-bg$, $H=fg-ch$.

For the conic (4), $A=-1/b^2$, $B=-[(1+\lambda)/a^2]$, $C=[(1-\lambda)/(a^2b^2)]$, $F=-\lambda/a^2b^2$, $G=H=0$, and these in (6) and (7) give

$$(1-\lambda)(x^2-y^2)-2b\lambda y+b^2(1-\lambda)-a^2=0 \ldots\ldots (8),$$

$$(1-\lambda)y+b\lambda=0 \ldots\ldots (9).$$

Eliminating λ from (8) and (9), the required locus is

$$x^2+y^2+\frac{a^2-2b^2}{b}y=a^2-b^2 \quad \ldots\ldots (10).$$

This is a circle, radius $a^2/2b$, or one-half the radius of curvature of (1) at the extremity of the minor axis.

115. Proposed by MARY M. BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

The locus of a point such that the sum of the squares of its normals form a given ellipsoid is constant, is a co-axial ellipsoid. [From *C. Smith's Solid Analytical Geometry*, page 95.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let (l, m, n) be the point whose locus is required. Then from the conditions of the problem we have

$$(l-x_1)^2+(l-x_2)^2+(l-x_3)^2+(l-x_4)^2+(l-x_5)^2+(l-x_6)^2$$
$$+(m-y_1)^2+(m-y_2)^2+(m-y_3)^2+(m-y_4)^2+(m-y_5)^2+(m-y_6)^2$$
$$+(n-z_1)^2+(n-z_2)^2+(n-z_3)^2+(n-z_4)^2+(n-z_5)^2+(n-z_6)^2=C^2.$$

This equation can be written more briefly thus :

$$\Sigma(l-x_2)^2+\Sigma(m-y_1)^2+\Sigma(n-z_1)^2=C.$$

$$\therefore C(l^2+m^2+n^2)-2[l\Sigma(x_1)+m\Sigma(y_1)+n\Sigma(z_1)]$$
$$+\Sigma(x_1^2)+\Sigma(y_1^2)+\Sigma(z_1^2)=C^2 \ldots\ldots (1).$$

The equation of the normal is

$$(a^2/x)(l-x)=(b^2/y)(m-y)=(c^2/z)(n-z) \ldots\ldots (2).$$

Also, $x^2/a^2+y^2/b^2+z^2/c^2=1 \ldots\ldots (3)$.

The values of y, z in terms of x from (2) in (3) gives

$$x^2/a^2+\frac{b^2m^2x^2}{(a^2l-a^2x+b^2x)^2}+\frac{c^2n^2x^2}{(a^2l-a^2x+c^2x)^2}=1\ldots\ldots(4).$$

(4) can be written as follows :

$$x^6+Ax^5+Bx^4+Dx^3+Ex^2+Fx+G=0.$$

Then $\Sigma(x_1)=-A$, $\Sigma(x_1x_2)=B$, $\Sigma(x_1^2)=A^2-2B$.

But $A=-\frac{2a^2l}{a^2-c^2}-\frac{2a^2l}{a^2-b^2}$.

$$B=\frac{a^2(a^2l^2+c^2n^2)}{(a^2-c^2)^2}+\frac{a^2(a^2l^2+b^2m^2)}{(a^2-b^2)^2}+\frac{4a^4l^2}{(a^2-b^2)(a^2-c^2)}-a^2.$$

$$\therefore \Sigma(x_1)=\frac{2a^2l}{a^2-c^2}+\frac{2a^2l}{a^2-b^2}.$$

$$\Sigma(x_1^2)=\frac{2a^2(a^2l^2-c^2n^2)}{(a^2-c^2)^2}+\frac{2a^2(a^2l^2-b^2m^2)}{(a^2-b^2)^2}+2a^2.$$

By symmetry,

$$\Sigma(y_1)=\frac{2b^2m}{b^2-c^2}+\frac{2b^2m}{b^2-a^2}.$$

$$\Sigma(y_1^2)=\frac{2b^2(b^2m^2-c^2n^2)}{(b^2-c^2)^2}+\frac{2b^2(b^2m^2-a^2l^2)}{(b^2-a^2)^2}+2b^2.$$

$$\Sigma(z_1)=\frac{2c^2n}{c^2-b^2}+\frac{2c^2n}{c^2-a^2}.$$

$$\Sigma(z_1^2)=\frac{2c^2(c^2n^2-b^2m^2)}{(c^2-b^2)^2}+\frac{2c^2(c^2n^2-a^2l^2)}{(c^2-a^2)^2}+2c^2.$$

Substituting these values in (1) and reducing we get

$$6(l^2+m^2+n^2)+\frac{2(c^2n^2-a^2l^2)}{a^2-c^2}+\frac{2(b^2m^2-a^2l^2)}{a^2-b^2}+\frac{2(c^2n^2-b^2m^2)}{b^2-c^2}$$
$$=C^2-2(a^2+b^2+c^2).$$

$$\therefore \frac{2(a^4-2a^2b^2-2a^2c^2+3b^2c^2)l^2}{(a^2-c^2)(a^2-b^2)(C^2-2a^2-2b^2-2c^2)}$$

$$+\frac{2(2b^2c^2+2a^2b^2-3a^2c^2-b^4)m^2}{(a^2-b^2)(b^2-c^2)(C^2-2a^2-2b^2-2c^2)}$$

$$+\frac{2(c^4-2a^2c^2-2b^2c^2+3a^2b^2)n^2}{(a^2-c^2)(b^2-c^2)(C^2-2a^2-2b^2-2c^2)}=1.$$

$\therefore\ l^2/R^2+m^2/S^2+n^2/T^2=1$, a co-axial ellipsoid.

CALCULUS.

89. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Integrate the equation, $\frac{dy}{dx}+y\cos x=\frac{\sin 2x}{2}$.

I. Solution by Dr. E. D. ROE, Jr., Norwood, Mass.

Put $\sin x-y=z$; then $\cos x dx-dy=dz$, $dy=\cos x dx-dz$, and the equation becomes

$$\frac{dz}{dx}+(z-1)\cos x=0, \text{ or } \frac{dz}{z-1}+\cos x dx=0.$$

Integrating this, $\log(z-1)+\sin x+\varkappa=0$, $z-1=e^{-\sin x-\varkappa}$,

$$\text{or } \sin x-y-1=e^{-\sin x-\varkappa}=-ce^{-\sin x},\ y=\sin x-1+ce^{-\sin x}.$$

Dr. Roe also furnished a second solution.

II. Solution by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.

If the equation $\frac{dy}{dx}+y\cos x=0$ is solved, the result is $y=C_1e^{-\sin x}$. After substituting in the original equation $\frac{dC_1}{dx}$ is found to equal $e^{\sin x}\sin x\cos x$; therefore, $C_1=e^{\sin x}\sin x-e^{\sin x}+C$. And $y=\sin x+Ce^{-\sin x}$.

III. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass., and JOHN R. JEFFERY, Student in Ohio State University, Columbus, O.

The general form of this equation is $\frac{dy}{dx}+Py=Q$ of which the general integral is $e^{\int Pdx}y=\int e^{\int Pdx}Qdx+C$. Here $\int Pdx=\int\cos x dx=\sin x$.

$\therefore\ e^{\sin x}y=\int e^{\sin x}\cos x\sin x dx+C$.

Integrating right member by parts, $e^{\sin x}y=\sin x.e^{\sin x}-e^{\sin x}+c$, or $y=\sin x-1+ce^{-\sin x}$.

[See Johnson's *Differential Equations*, page 35, ex. 7.]

IV. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics in Manual Training School, Philadelphia, Pa.; M. C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Ind.; and BURKE SMITH, Senior Class, University of Washington, Seattle, Wash.

$\frac{dy}{dx}+y\cos x=\sin x\cos x.$

Multiply by $e^{\sin x}$ as an integrating factor.

Then $e^{\sin x}dy+y\cos x.e^{\sin x}dx=e^{\sin x}\sin x\cos x dx.$

Integrating, $y.e^{\sin x}=\int e^{\sin x}\sin x\cos x dx.$

Integrate right hand member by parts, and we have,

$$ye^{\sin x}=e^{\sin x}\sin x-\int e^{\sin x}\cos x dx=e^{\sin x}\sin x-e^{\sin x}+c.$$

$\therefore y=\sin x-1+ce^{-\sin x}.$

V. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $\sin x=\log v$; so that $\cos x dx=dv/v$. Then the equation becomes $dy/dv+y/v=\log v/v$.

$\therefore vdy+ydv=\log v dv. \quad \therefore vy=v(\log v-1)+C. \quad \therefore y+1=\log v+C/v.$

$\therefore y+1=\sin x+Ce^{-\sin x}.$

VI. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and ARTHUR C. LUNN, University of Chicago, 5630 Drexel Ave., Chicago, Ill.

Solving first the differential equation $\frac{\partial y}{\partial x}+y\cos x=0$, we find $y=Ce^{-\sin x}$; differentiating, we have $\frac{\partial y}{\partial x}=-Ce^{-\sin x}\cos x+e^{-\sin x}\frac{\partial C}{\partial x}.$

$\therefore e^{-\sin x}\frac{\partial C}{\partial x}=\frac{1}{2}\sin 2x. \quad \therefore C=\frac{1}{2}\int e^{\sin x}\sin 2x.dx.$

Putting $\sin x=z$, $\therefore \cos x=\sqrt{1-z^2}$, $\partial x=\frac{\partial z}{\sqrt{1-z^2}}$; we have,

$$C=\int e^z.z\partial z=\int z\partial(e^z)=ze^z-\int e^z\partial z=ze^z-e^z=\sin x.e^{\sin x}-e^{\sin x}+c.$$

Substituting this in $y=Ce^{-\sin x}$, we obtain $y=\sin x-1+ce$.

The method employed is Lagrange's method of variation of parameters.

Also solved by *HENRY HEATON*, and *P. H. PHILBRICK*.

MECHANICS.

82. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A sphere, diameter $2a$, rests in limiting equilibrium upon the edge of a box and against a vertical wall. If the box be of such dimensions that it will not tip, find the distance of the box from the wall, having given the coefficient of friction between the sphere and wall $\frac{1}{2}$, between the sphere and box $\frac{1}{3}$, and between the box and floor $\frac{2}{3}$. [From Problems in Mechanics proposed to class in Harvard University.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let W=weight of sphere, W'=weight of box, $\mu=\frac{1}{2}$, $\mu'=\frac{1}{3}$, $\mu''=\frac{2}{3}$, $\theta=\angle BCD$, S=normal reaction of wall, R=normal reaction of box, d=distance of box from wall.

$\therefore\ d=AO+BE=a(1+\sin\theta)$, since BO is perpendicular to BC.

Also $S=\mu' R\cos\theta+R\sin\theta=\mu'' W'$ (resolving horizontally).

$$\therefore\ S=\tfrac{2}{3}W',\ R=\frac{\mu'' W'}{\mu'\cos\theta+\sin\theta}$$

$$=\frac{2W'}{\cos\theta+3\sin\theta}.$$

Also $\mu S+\mu' R\sin\theta+R\cos\theta=W$ (resolving vertically), or $\frac{1}{2}S+\frac{1}{3}R\sin\theta+R\cos\theta=W$.

The values of S and R in the last equation give

$$\tfrac{1}{3}W'+\frac{2W'\sin\theta}{3\cos\theta+9\sin\theta}+\frac{2W'\cos\theta}{\cos\theta+3\sin\theta}=W.$$

$$\therefore\ \tan\theta=\frac{7W'-3W}{9W-5W'},\ \sin\theta=\frac{7W'-3W}{\sqrt{90W^2-132WW'+74W'^2}}.$$

$$\therefore\ d=a\left[\frac{\sqrt{90W^2-132WW'+74W'^2}+7W'-3W}{\sqrt{90W^2-132WW'+74W'^2}}\right].$$

If $W=W'$, $d=\frac{1}{2}a(2+\sqrt{2})$.

83. Proposed by MARY M. BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

A particle is projected upwards in vacuo with a velocity v. Show that on reaching the ground again there is no deviation to the south, but the deviation to the west is $4\omega\cos\lambda(v^3/3g^2)$. [*Laplace*, iv, page 341.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The equations of motion are

$$\frac{d^2x}{dt^2}+2\omega\sin\lambda\frac{dy}{dt}=0,\quad \frac{d^2y}{dt^2}-2\omega\cos\lambda\frac{dz}{dt}-2\omega\sin\lambda\frac{dx}{dt}=0,\quad \frac{d^2z}{dt^2}+2\omega\cos\lambda\frac{dy}{dt}=-g.$$

(Routh's *Advanced Rigid Dynamics*, fourth edition, page 20.)

As a first approximation, we can neglect the motion of the earth. Then, from mechanics, $x=0$, $y=0$, $z=vt-\frac{1}{2}gt^2$.

$$\therefore \frac{dx}{dt}=0,\quad \frac{dy}{dt}=0,\quad \frac{dz}{dt}=v-gt.$$

$$\therefore \frac{d^2x}{dt^2}=0,\quad \frac{d^2y}{dt^2}=2\omega\cos\lambda(v-gt).$$

$\therefore x=0$, $y=\omega\cos\lambda(vt^2-\frac{1}{3}gt^3)$. But $t=2v/g$.

$\therefore x=0$, $y=\omega\cos\lambda(4v^3/g^2-8v^3/3g^2)=4\omega\cos\lambda(v^3/3g^2)$.

$\therefore$ deviation south$=0$, west$=4\omega\cos\lambda(v^3/3g^2)$.

AVERAGE AND PROBABILITY.

67. Proposed by HENRY HEATON, M. Sc., Atlantic, Ia.

A witness in court who undertook to recognize the signature of an individual failed four times in succession. What is the probability that he was correct the fifth time? An actual occurrence.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and CHARLES CARROLL CROSS, Libertytown, Md.

Let p=chance, p_1=chance of failure.

$$\therefore p_1=\frac{\int_0^1 x^5dx}{\int_0^1 x_4dx}=\frac{5}{6}.\quad \therefore p=1-\frac{5}{6}=\frac{1}{6}.$$

68. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

What are the odds against throwing 7 or 11 at one throw with two dice?

I. Solution by CHARLES CARROLL CROSS, Libertytown, Md.

Each dice may fall in any one of 6 ways, therefore, both dice in $6\times6=36$ ways.

$11=6+5$ or $5+6$; hence the chance against throwing 11 at one throw is $1-\frac{2}{36}=\frac{17}{18}$.

$7=1+6=2+5=3+4=4+3=5+2=6+1$; hence the chance against throwing 7 at one throw is $1-\frac{6}{36}=\frac{5}{6}$.

Hence, the chance against throwing either 7 or 11 is $1-(\frac{1}{18}+\frac{1}{6})=\frac{7}{9}$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

The chance of throwing 7 with two dice is $\frac{6}{36}=\frac{1}{6}$, and that of throwing 11 is $\frac{2}{36}=\frac{1}{18}$; therefore, the chance of throwing either 7 or 11 is $\frac{1}{6}+\frac{1}{18}=\frac{2}{9}$.

$\therefore$ the odds against this event$=7:2$.

III. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

The combinations which give 7 or 11 with two dice are four in number, viz.: $6+1$, $5+2$, $4+3$, and $6+5$.

The total number of combinations is 36. Hence the chance of throwing either a 7 or 11 at one throw is $\frac{4}{36}=\frac{1}{9}$. Hence the odds are 8 to 1 against the event.

IV. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

7 can be thrown 6 ways at one throw with two dice, as follows:

6 and 1, 1 and 6, 5 and 2, 2 and 5, 4 and 3, 3 and 4.

$\therefore$ chance of throwing 7 is $\frac{6}{36}=\frac{1}{6}$.

$\therefore$ the odds are 5 to 1 against the event.

11 can be thrown 2 ways, as follows: 6 and 5, and 5 and 6.

$\therefore$ chance of throwing 11 is $\frac{2}{36}=\frac{1}{18}$.

$\therefore$ the odds are 17 to 1 against the event.

7 is the most likely throw of all at one time with two dice.

V. Solution by ELMER SCHUYLER, Annapolis, Md.

$7=1+6=2+5=3+4$, each occuring in two ways.

$\therefore$ 7 can occur in six ways.

$11=5+6$, can occur in 2 ways.

$\therefore$ 7 and 11 can occur in eight ways.

$\therefore \frac{8}{36}=\frac{2}{9}=$probability in favor of the event.

$\therefore$ the odds are 7 to 2.

69. Proposed by Rev. W. ALLEN WHITWORTH, M. A.

There are n equal sugar sticks. Each stick is broken into two pieces, all positions of the fracture being equally likely. Of the two n pieces thus formed, a child is to take the largest. Show that his expectation is $[2n+1]/[2(n+1)]$ of a stick. [From *The Educational Times*, June, 1898.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let each stick be of unit length. Then since there are $2n$ pieces, and each piece can have any length from zero to unity, we have for the required expectation:

$$E=\frac{\int_0^1 \sqrt{x^{2n+1}}dx}{\int_0^1 x^{2n}dx}=\frac{2n+1}{2(n+1)}.$$

70. Proposed by Professor MILLER.

A ship at A observes another at B, whose course is unknown. Supposing their speed the same, prove that the chance of their coming within a given distance, d, of each other is always $(2/\pi)\sin^{-1}(d/a)$, whatever the course taken by A; provided its inclination to AB is not greater than $\cos^{-1}(d/a)$, where $AB=a$. [From *Cambridge Mathematical Tripos*, 1871.]

No solution of this problem has been received.

71. Proposed by B. F. FINKEL, A.M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the average volume removed by boring an inch auger-hole through a cube whose edge is e, the auger to pass through two opposite faces of the cube.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The average volume required is equal to the altitude multiplied by the average base. All variations of the base will be gone through by considering a cylinder through the center. The greatest base is the ellipse formed when the cylinder is tangent to two edges of the cube. Its axis is then determined by the equations $y-z\tan\beta=0$.....(1), and $x-z\tan\beta=0$.....(2).

β is the angle made by the plane through the origin, and the X-axis with the Y-axis. This plane is parallel to the edge $y=\frac{1}{2}e$, $z=\frac{1}{2}e$, and at a distance of $\frac{1}{2}$ inch from this edge.

$\therefore \sin\beta=\sin(\frac{1}{2}\pi-\theta)$, where $\sin\theta=(\sqrt{2}/2e)$, as follows:

$$\sin\theta : 1=\tfrac{1}{2} : \tfrac{1}{2}(e\sqrt{2}).$$

The line given by

$$\left.\begin{aligned} y\cos\beta-z\sin\beta&=-\tfrac{1}{2}\\ x\cos\beta-z\sin\beta&=-\tfrac{1}{2}\end{aligned}\right\}\ .....(3).$$

x, y from (1), (2), (3), when $z=\frac{1}{2}e$ are

$$x_1=\tfrac{1}{2}e\tan\beta,\ y_1=\tfrac{1}{2}e\tan\beta, \text{ and } x_2=y_2=-\tfrac{1}{2}e\sec\beta+\tfrac{1}{2}e\tan\beta.$$

$$\therefore \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\frac{e\sec\beta\sqrt{2}}{2}=\text{semi-major axis}.$$

The semi-minor axis $\frac{1}{2}$ inch for all ellipses

$$e\sec\beta=\frac{1}{\cos\beta}=\frac{1}{\cos(\frac{1}{2}\pi-\theta)}=\frac{\sqrt{2}}{\cos\theta+\sin\theta}=\frac{2e}{\sqrt{(2e^2-1)}+1}.$$

$\therefore \sec\beta = \frac{e[\sqrt{(2e^2-1)}-1]}{e^2-1}$. $\therefore \frac{e\sqrt{(2)}[\sqrt{(2e^2-1)}-1]}{2(e^2-1)}$ is semi-major axis.

$\therefore$ average volume is $\frac{1}{3}e\left(\frac{e\sqrt{(2)}[\sqrt{(2e^2-1)}-1]}{2(e^2-1)}+\frac{1}{2}\right)\frac{1}{2}\pi = \triangle$.

$\therefore \triangle = \frac{\pi e}{8(e^2-1)}[e\sqrt{2}(\sqrt{2e^2-1}-1)+e^2-1] = 5.4345$ cubic inches, when $e=5$ inches.

72. Proposed by B. F. FINKEL, A.M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A rod is broken at random into four pieces; find the chance that no one of the pieces is greater than the sum of the other three. [From *C. Smith's Treatise on Algebra*, p. 528.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let a=length of rod.

By conditions of problem no part can be greater than $\frac{1}{2}a$.

Let $ABCD-G$ be a cube side a.

Let $Abcd-g$ be a cube side $\frac{1}{2}a$.

For favorable cases the points are confined to the smaller cube.

$\therefore$ chance$= \frac{(\frac{1}{2}a)(\frac{1}{2}a)(\frac{1}{2}a)}{(a)(a)(a)} = \frac{1}{8}$.

73. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

On an average 1 vessel out of every n is wrecked. Find the chance that out of m vessels expected p at least will arrive safely.

I. Solution by the PROPOSER.

The chance of a vessel arriving is $[(n-1)/n]$.

The chance of a vessel not arriving is $1/n$.

The event will happen if, m, $(m-1)$, $(m-2)$, $(m-3)$, $(m-4)$, ... down to p vessels arrive.

Thus the required chance is the sum of the first $(m-p+1)$ terms in the expansion of

$$\left(\frac{n-1}{n}+\frac{1}{n}\right)^m = \left(\frac{n-1}{n}\right)^m + \frac{m}{1}\left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^{m-1} + \frac{m(m-1)}{2!}\left(\frac{1}{n}\right)^2\left(\frac{n-1}{n}\right)^{m-2}$$

$$+ \cdots\cdots \frac{m!}{p!(m-p)!}\left(\frac{1}{n}\right)^{m-p}\left(\frac{n-1}{n}\right)^p.$$

If $n=10$, $m=5$, $p=3$, we get chance$= (\frac{9}{10})^5 + 5(\frac{1}{10})(\frac{9}{10})^4 + 10(\frac{1}{10})^2(\frac{9}{10})^3 = \frac{1}{100}\frac{8}{0}\frac{3}{0}\frac{8}{0}$.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Ia.

For convenient notation put $n-1=k$.

Then the probability that any given vessel will be wrecked is $1/n$, that it will be saved, k/n.

The probability that all the m vessels will be saved is k^m/n^m.

The probability that any particular vessel will be lost while the others are saved is $(k/n)^{m-1}(1/n)$.

There are m vessels. Hence the probability that *one* vessel will be lost while the others are saved is $m(k^{m-1}/n^m)$.

The probability that any two particular vessels will be lost while the others are saved is $(k/n)^{m-2}(1/n)^2$.

The number of different combinations for two vessels to be saved and the rest lost is $\frac{m(m-1)}{2!}$.

Hence the probability that two vessels will be lost is $\frac{m(m-1)}{2!}\frac{k^{m-2}}{n^m}$.

In like manner the probability that 3 vessels will be lost and $m-3$ saved is $\frac{m(m-1)(m-2)k^{m-3}}{n^m 3!}$ and that 1, 2, 3, ... or $m-p$ vessels will be lost is

$$P=\frac{1}{n^m}\left(k^m+mk^{m-1}+\frac{m(m-1)k^{m-2}}{2!}\cdots\cdots+\frac{m(m-1)\cdots\cdots(p-1)\,k^p}{(m-p)!}\right).$$

If $p=m$, $P=\left(\frac{k}{n}\right)^m$. If $p=0$, $P=\frac{(k+1)^m}{n^m}=1$, as it evidently should.

III. Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

The chance that any particular ship arrives is $\frac{n-1}{n}$. The chance that any particular set of p ships arrives is $\left(\frac{n-1}{n}\right)^p$; and the chance that the other $m-p$ ships fails is $\left(\frac{1}{n}\right)^{m-p}$. Hence the chance that any particular set of p ships alone arrives is $\left(\frac{n-1}{n}\right)^p\left(\frac{1}{n}\right)^{m-p}$. But out of m ships p can be selected in $\frac{m(m-1)\cdots\cdots(m-p+1)}{1.2\cdots\cdots p}$ ways. Hence the chance that some possible set of p ships will arrive is

$$\left(\frac{n-1}{n}\right)^p\left(\frac{1}{n}\right)^{m-p}\frac{m(m-1)\cdots\cdots(m-p+1)}{1.2\cdots\cdots p}\cdots\cdots(1).$$

Writing $p+1$, $p+2$, $\ldots\ldots m$, successively, in place of p in (1) will give

the chances that $p+1$, $p+2$, ... m ships will arrive. The sum of these added to (1) gives the chance that *at least* p ships out of m will arrive.

Also solved by *ELMER SCHUYLER*.

MISCELLANEOUS.

71. Proposed by GUY B. COLLIER, 1901 Union, 27 Middle Section of South College, Schenectady, N. Y.

Find the locus of any point on the front sprocket of a bicycle during one revolution of the hind wheel (any gear may be assumed).

I. Solution by R. E. GAINES, A. M., Professor of Mathematics, Richmond, Va.

Suppose the front sprocket revolves e times as fast as the hind wheel of the bicycle (if, to fix the idea, the gear be 70, then $e=\frac{2}{5}$), and take as axis of x the line on which the bicycle rolls, and as axis of y a line perpendicular to this, and passing through P when it was in its lowest position. Then let the hind wheel turn through an angle φ (the sprocket turning through an angle $e\varphi$) and the *center* of the sprocket has moved forward a distance $a\varphi$ and we obtain at once

$$x=a\varphi-r\sin(e\varphi), \quad y=a-r\cos(e\varphi),$$

where r=radius of sprocket, a=radius of hind wheel, and φ=angle through which the hind wheel has turned. Expressed as a single equation, this is

$$x=(a/e)\cos^{-1}\frac{a-y}{r}-r\sqrt{1-\left(\frac{a-y}{r}\right)^2}.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let O be the origin, $OF=x$, $PF=y$, P=any point in the front sprocket, g=gear, $AD=a$, radius of hind wheel, $AB=b$, $BP=c$, and also let $BE=AD=a$, $\angle MAD=\theta$.

Then $BH/AN=g/2a$.

$\therefore \angle HBE=(2a/g)\theta$.

$\therefore x=OD+DE-FE=OD+AB-BC$.

$\therefore x=b+a\theta-c\sin(2a\theta/g)$.

$\therefore y=a-c\cos(2a\theta/g)$.

$\therefore x=b+(\frac{1}{2}g)\cos^{-1}\left(\frac{a-y}{c}\right)-1\sqrt{c^2-(a-y)^2}$ is the equation to the locus.

s=length for one revolution of hind wheel.

$$\therefore s=(1/g)\int_0^{2\pi}1\sqrt{a^2g^2+4a^2c^2-4a^2cg\cos(2a\theta/g)}\,d\theta$$

$$=(a/g)\int_0^{2\pi}\sqrt{(g+2c)^2-8cg\cos^2(a\theta/g)}\,d\theta.$$

Let $\frac{1}{2}\pi-\psi=a\theta/g$.

$$\therefore\ s=(g+2c)\int_{(\pi/2g)(g-4a)}^{\frac{1}{2}\pi}\sqrt{1-\frac{8cg}{(g+2c)^2}\sin^2\psi}\,d\psi$$

$$=(g+2c)E^{\frac{1}{2}\pi}_{(\pi/2g)(g-4a)}\left(\frac{2\sqrt{(2cg)}}{g+2c},\ \psi\right).$$

Also solved by *WALTER H. DRANE*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

118. Solution by J. F. TRAVIS, Student in Ohio State University, Columbus, O.

The present worth of a note due January 1, 1896, was \$74,200, when discounted at 4% true discount. The present worth of another note, due July 1, 1896, whose face value was the same as that of the first note, was \$68900 when discounted at 8% true discount. Find the face of the notes and the date when given, supposing the second note to have been given the same day the first note was. Solve by arithmetic.

119. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is \$1.$\frac{9}{00}$. The selling price \$1,000. What is the gain per cent.?

*** Solutions of these problems should be sent to B. F. Finkel not later than November 10.

ALGEBRA.

107. Proposed by CHARLES MYERS, Canton, Ohio.

Given $xyz=18\ldots.(1)$; $x^2+y^2+z^2=33\ldots.(2)$, and $(x^2-yz)^3+(y^2-xz)^3+(z^2-xy)^3-3(x^2-yz)(y^2-xz)(z^2-xy)=6561\ldots.(3)$, to find x, y, and z.

108. Proposed by GEORGE LILLEY, Ph.D., LL.D., Professor of Mathematics, State University, Eugene, Or.

A gave two notes; one for a dollars at m per cent., and the other for b dollars at n per cent., annual interest. He is to make a monthly payment of c dollars. How much must be endorsed on each note in order to pay them off at the same time? What must be the endorsement on each if $a=1900$, $b=1800$, $m=6$, $n=7$, and $c=25$?

*** Solutions of these problems should be sent to J. M. Colaw not later than November 10.

GEOMETRY.

126. Proposed by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Through any fixed point O draw two straight lines at right angles. Let one line cut a given circle at Q, the other at R. Find, by Euclidean methods, the locus of the foot of the perpendicular from O upon the chord QR. Give complete analysis and discussion. Solve also by coördinate geometry.

127. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

The equation to the plane through the extremities, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), of conjugate diameters of the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is } \frac{x_1+x_2+x_3}{a^2}x+\frac{y_1+y_2+y_3}{b^2}y+\frac{z_1+z_2+z_3}{c^2}z=1.$$

*** Solutions of these problems should be sent to B. F. Finkel not later than November 10.

CALCULUS.

97. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

An augur hole radius r is bored through a prolate spheroid; the axis of the augur passing through the center, perpendicular to the major axis. Find the volume removed.

98. Proposed by CHARLES CARROLL CROSS, Whaleyville, Va.

On the circumference of a fixed circle radius R rolls a circle radius r. Required the length of the curve described by a point on the circumference of the rolling circle; (1) when the circle rolls on the inside; (2) when the circle rolls on the outside of the circumference of the fixed circle.

*** Solutions of these problems should be sent to J. M. Colaw not later than November 10.

MECHANICS.

97. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The side AB of the parallelogram $ABCD$ will be a principal axis at the point which divides the distance between the middle point and the foot of the perpendicular from the middle-point of the opposite side in the ratio 2 : 1. The principal moments of inertia about this point are $\frac{1}{3}mb^2\sin^2\beta$, $\frac{1}{36}m(3a^2+4b^2\cos^2\beta)$, where $\beta=\angle A$.

98. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A spool, with light thread wound around, is placed upon a rough table so that the thread will emerge from beneath the spool. The thread is passed over a smooth pulley at end of table and a weight attached, the pulley being so adjusted that thread is parallel to surface of table. If friction between spool and table is sufficient to prevent slipping, de-

termine motion of spool and weight. [From problems in Mechanics at Harvard University.]

*** Solutions of these problems should be sent to B. F. Finkel not later than November 10.

AVERAGE AND PROBABILITY.

80. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A box contains 100 balls marked from 1 to 100. 13 balls are drawn at random. What is the chance that the balls marked from 1 to 10 are included in the 13 drawn?

81. Proposed by LON C. WALKER, Graduate of Leland Stanford, Jr., University, Palo Alto, Cal.

Find (1) the mean distance of all points on a side of an equilateral triangle from the opposite vertex; and (2), the average length of a line drawn at random across an equilateral triangle.

*** Solutions of these problems should be sent to B. F. Finkel not later than November 10.

MISCELLANEOUS.

81. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A cask in the form of a middle frustum of a spheroid, middle diameter $2b$, end diameters each $2c$, length $2d$, is lying in a horizontal position. The distance from middle of top to water is $b+e$, $e<b-c$. How much water is in the cask?

82. Proposed by A. H. BELL, Hillsboro, Ill.

Four spheres of equal radii $=r=5$, are in contact, and form a triangular period. How large is the sphere that can be placed in middle and be in contact with the four spheres?

*** Solutions of these problems should be sent to J. M. Colaw not later than November 10.

EDITORIALS.

Prof. J. O. Mahoney has been elected teacher of mathematics in the Dallas High School, Dallas, Texas.

Mr. Peter Field, Fellow in Cornell University, has been appointed Professor of Mathematics in Carthage College, Carthage, Ill.

Dr. J. V. Westfall, Honorary Fellow in Cornell University, has been appointed Senior Instructor in Mathematics in the Iowa State University.

Miss Mary M. Blaine, B. S. (Drury College), has been given a scholarship in the University of Pennsylvania, and has gone there to pursue a course of study in mathematics.

The Franklin Institute of Philadelphia has lately created the office of mathematical contributor to the Physical and Astronomical Section, and has appointed Dr. G. B. M. Zerr as the contributor for the coming year.

This issue of this MONTHLY has been unavoidably delayed. Subsequent issues will be out on time. The next issue will contain more than the usual number of pages, as an article of unusual interest, by Dr. Halsted, is to appear.

Dr. L. E. Dickson, formerly Assistant Professor of Mathematics in the University of California, has been elected Associate Professor of Mathematics in the University of Texas. Dr. Halsted is to be congratulated in being able to call to his assistance one of the ablest young mathematicians in the country. With Dr. Halsted noted for his valuable work in non-Euclidean Geometry, and Dr. Dickson a recognized authority on Group Theory, the University of Texas will be able to offer as good courses in mathematics as are offered anywhere in this country.

The Sixth Summer Meeting of the American Mathematical Society met at Columbus, Ohio, August 25th and 26th. The meeting was well attended, and the papers read and discussed were of great value. Dr. J. V. Collins read a paper on "A relation between point and vector analysis"; Dr. E. H. Moore presented Prof. Frank Morley's paper, "On the generalization of Desargues' theorem," and one of his own "On certain crinkly curves"; Dr. G. A. Miller read a paper "On groups that are the direct products of two subgroups"; and Dr. L. E. Dickson read two papers, one on "A new definition of the general Abelian linear group," the other on "Definitions of various linear groups as groups of isomorphisms." Dr. Halsted's able report on Non-Euclidean Geometry was read before Section A of the Association for the Advancement of Science on Tuesday, and was greatly appreciated by his audience. Dr. Alexander Macfarlane, president of Section A, read a very interesting paper on "The fundamental principles of algebra." Dr. Macfarlane gave a very interesting and exhaustive treatment of this subject, tracing the important advances in the philosophy of the fundamental principles of algebra which have been made in the present century.

At this meeting we had the pleasure of meeting a number of our good friends. Among those, whom we had long known through correspondence, but now have the pleasure of knowing personally, were Dr. Halsted, of the University of Texas ; Prof. Ormond Stone, of the University of Virginia ; Prof. R. S. Woodward, of Columbia ; Prof. D. V. Bohannon, of the Ohio State University ; Dr. J. V. Collins, of the State Normal School of Wisconsin ; Prof. W. W. Beman, of the University of Michigan ; Mr. J. W. Young, Graduate Student, Ohio State University ; Dr. Alexander Macfarlane, Lehigh University ; and Prof. F. E. Miller, of Otterbein University.

BOOKS AND PERIODICALS.

Stereoscopic Views of Solid Geometry, with Reference to Wells' Essentials of Solid Geometry. Set, 96 views. Price, 60 cents per set. Boston: D. C. Heath & Co.

These views are very beautiful and attractive. They are printed on heavy card board, having a black background and white lines. B. F. F.

A Treatise on the Theory and Solutions of Algebraical Equations. By John Macnie, M. A., Professor of the Latin Language and Literature, University of North Dakota. Large 8vo. Half Leather Back. 184 pages. Price, $1.20, net. Chicago: A. S. Barnes & Co.

This work discusses, in a very careful and accurate way, Cauchy's Theorem, Sturm's Theorem, Horner's Method, Fourier's Theorem, Symmetric Functions of the Roots, Elimination, etc., etc. B. F. F.

New Higher Algebra. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 8vo. Half Leather Back. 446 pages. Price, $1.32. Boston: D. C. Heath & Co.

The first 358 pages of this book are identical with the corresponding pages of the author's *Essentials of Algebra*. The new matter added is Compound Interest and Annuities, Permutations and Combinations, Continued Fractions, Summation of Series, Theory of Equations, and Solution of Higher Equations. B. F. F.

An Elementary Treatise on the Theory of Equations. By Samuel Marx Barton, Ph. D., Professor of Mathematics, University of the South. 8vo. Cloth, 200 pages. Price, $1.50. Boston: D. C. Heath & Co.

In this book is set forth the elements of Determinants and the Theory of Equations in quantity and in form suitable for use in undergraduate work in our colleges and universities. Part I treats of Determinants, and Part II of the Theory of Equations. The book is well adapted to the purpose for which it was written. B. F. F.

Memoir on the Substitution Groups whose Degree does not Exceed Eight. By Dr. G. A. Miller, of Cornell University. Reprinted from *American Journal of Mathematics*, Vol. XXI, No. 4, pages 287—338.

In this memoir Dr. Miller has discussed the Substitution Groups whose degree does not exceed eight. Dr. Miller, though a young man, is one of the leading authorities on Group Theory in this country. His contributions on this subject are characterized by accuracy and simplicity of treatment. This memoir is one of great value to the student of Groups, and will become a permanent part of the literature on the subject. B. F. F.

A Report on Greene County. Part I. of Geological Survey of Missouri, by Edward M. Shepard, M. A., Professor of Geology in Drury College. 8vo. Cloth. 246+iv pages. Printed by the State of Missouri.

Among the interesting points about this book may be mentioned the fact that it is wholly a pioneer work, and that the author has discovered and named a number of geological formations hitherto unknown. Among these are the "Republic chert" and the "Graydon sandstone" and its connection with an ancient river system, notably the fossil river bed which he has named the "Schoolcraft river," extending through the whole western half of the district and northward to the Missouri river, into which its ancient waters

poured. He was the first to discover the presence of Devonian rocks in Southwest Missouri, from which he has named three horizons new to science, viz., the Phelps sandstone, the Sac limestone, and the King limestone. He was also the first to demonstrate the presence of the great flexuring and faulting in the Ozark uplift. The discussion of the ore deposits of the district is particularly valuable, and the large list of locations where lead and zinc have been found in the past will be of great interest to prospectors in this region.

Professor Shepard's extensive travels in all parts of the United States, in Cuba, in the Hawaiian Islands, in Australia, in New Zealand, and many other places, has enabled him to compare the region of which he has written with those of the highest geological interest. The book has many fine illustrations and a number of excellent geological maps.

B. F. F.

The Fundamental Principles of Algebra. An Address by Dr. Alexander Macfarlane, Vice President and Chairman of Section A of the Association for the Advancement of Science, delivered at Columbus, Ohio, August 21, before Section A of the Association. Pamphlet, 32 pages.

In this address Dr. Macfarlane reviews historically and critically the advances that have been made respecting the fundamental principles of Algebra. The work done along this line by Servais, Babbage, Herschel, Peacock, and others, is brought out. Mention is made of the elaborate treatise on Algebra by G. Chrystal, and several pages are devoted to a review of Mr. Whitehead's Treatise on Universal Algebra. The address should be read by everyone of our readers not having had the privilege of hearing it. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year, in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

A magazine of the highest literary and artistic merit. Read Mark Twain's article on "Christian Science and the Book of Mrs. Eddy" in the October number, and enjoy the pleasure of a good hearty laugh. B. F. F.

The Open Court. A Monthly Magazine Devoted to the Science of Religion, the Religion of Science, and the Extension of the Religious Parliament Idea. Dr. Paul Carus, Editor. T. J. McCormack, Assistant Editor. Price, $1.00 per year. Chicago: The Open Court Publishing Co.

In the October number are a number of interesting articles relating to Germany.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. New York: The Review of Reviews Co.

The American Monthly Review of Reviews for September contains a remarkably attractive group of contributed articles. The timeliness of the subjects treated is seen by a glance at the table of contents. The war in the Philippines is summed up by John Barrett; the outcome of The Hague conference is set forth by W. T. Stead; the subject of trusts is discussed by George E. Roberts and by Henry Macfarland; Hezekiah Butterworth writes of "The Future Value of the New England Farm," while Prof. L. H. Bailey answers affirmatively the question, "Does Farming Pay?" Sylvester Baxter tells of the progress made by the State of Massachusetts in her public library system, and Gilbert K. Harroun describes the work of the Cuban Educational Association of the United States; a sketch of "The New Secretary of War" is contributed by Henry Macfarland, while Dr. William Hayes Ward writes of Colonel Ingersoll, and Erica Glenton of the late Grand Duke George of Russia. B. F. F.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. VI. OCTOBER, 1899. No. 10.

REPORT ON PROGRESS IN NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED.

[Read at Columbus Meeting of the American Association for the Advancement of Science, August 21—26.]

It marks an epoch in the history of mathematics that at a meeting of a great Association for the advancement of science there should be presented by invitation a Report on non-Euclidean geometry.

Its two creators, Lobachévski who mis-named it Imaginary Geometry, and Bolyai János, under the nobler name Science Absolute of Space, failed utterly while they lived, to win any appreciative attention for what is to-day justly honored as one of the profoundest advances of all time. The only recognition, the only praise of the achievement of Lobachévski ever printed in his lifetime was by Bolyai Farkas the father of his brilliant young rival, and appeared in a little book with no author's name on the title page, and which we have no evidence that Lobachévski ever saw, a little book so rare that my copy is probably the only one on the western continent.

When after more than forty years their names were rescued from oblivion by Baltzer and Hoüel in 1866, still envious Time gave them back only with an aspersion against the genuineness of their originality. A cruel legend tarnished still their fame so long delayed, so splendidly deserved.

Even when their creation had reached the high dignity of made the subject of courses of lectures for consecutive semesters at the University of Göttingen, yet on page 175 of the second impression of these lectures, 1893, we still find Felix Klein saying, "Kein Zweifel bestehen kann, dass Lobatscheffsky

sowohl wie Bolyai die Fragestellung ihrer Untersuchungen der Gaussischen Anregung verdanken."

It is a privilege to begin my Report by announcing the rigorous demonstration that this ungenerous legend is untrue. This point need not further delay us, since it has been treated by me at length in *Science*, N. S. Vol. IX., No. 232, pages 813—817, June 9, 1899.

What a contrast to the pathetic neglect of its creators, Lobachévski dying blind, unrecognized, without a single follower, Bolyai János dying of disgust with himself and the world, lies in the fact that less than a year ago our American magazine the *Monist* secured from the famous Poincaré at great cost a brilliant contribution to this now universally interesting subject, which I had the honor, through my friend T. J. McCormack, of reading in the original French manuscript.

This extraordinary paper, published only in English translation, appears in the *Monist*, Vol. 9, No. 1, October, 1898, pages 1—43. In the first section of his greatest work Lobachévski says: "*Juxtaposition* (contact) is the distinctive characteristic of solids, and they owe to it the name *geometric solids*, when we retain this attribute, taking into consideration no others whether essential or accidental.

Besides bodies, for example, also time, force, velocity are the object of our judgment; but the idea contained in the word juxtaposition does not apply thereto. In our mind we attribute it only to solids, in speaking of their composition or dissection into parts.

This simple idea, which we have received directly in nature through the senses, comes from no other and consequently is subject to no further explanation. Two solids A and B, touching one another, form a single geometric solid C, in which each of the component parts A, B appears separate without being lost in the whole C. Inversely, every solid C is divided into two parts A and B by any *section* S.

By the word section we understand here no new attribute of the solid, but again a juxtaposition, expressing thus the partition of the solid into two juxtaposed parts.

In this way we can represent to ourselves all solids in nature as parts of a single whole solid which we call space."

Poincaré starts off somewhat differently. He says: "We at once perceive that our sensations vary, that our impressions are subject to change. The laws of these variations were the cause of our creating geometry and the notion of geometrical space.

Among the changes which our impressions undergo, we distinguish two classes:

(1) The first are independent of our will and not accompanied by muscular sensations. These are *external changes* so called.

(2) The others are voluntary and accompanied by muscular sensations. We may call these *internal changes*.

We observe next that in certain cases when an external change has modi-